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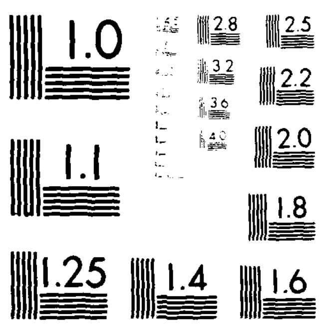

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We study the problem of an electromagnetic wave of the form $\vec{E} = (0,E(x,t),0)$, $\vec{B} = (0,0,B(x,t))$ propagating into the half-space $x > 0$, under the assumption that the half-space is occupied by a nonlinear dielectric material; the constitutive relations

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satisfied in the dielectric are of the form $\vec{D} = \epsilon(E)\vec{E}$, $\vec{B} = \mu(H)\vec{H}$ where $\epsilon > 0$, $\mu > 0$ are scalar-valued. By using Maxwell's equations we show that $\vec{E}$ and $\vec{H}$ satisfy a quasilinear system which is not in conservation form but that a quasilinear system of conservation type for $\vec{D}$ and $\vec{B}$ is naturally associated with the original system. Classical work of Lax for periodic initial fields and recent work of Majda and Klainerman for compactly supported initial fields imply the development of shock discontinuities in the electromagnetic field in the dielectric; the consequences of the Rankine-Hugoniot and Lax entropy conditions are computed for a nonmagnetic material with $\epsilon(E) = \epsilon_0 + \epsilon_2|\vec{E}|^2$ and $\mu(H) = \mu(\text{const})$. For such a material we also show that the nonzero component $D(x,t)$ of $\vec{D}$ satisfies the scalar nonlinear wave equation $\mu \frac{\partial^2 D}{\partial t^2} = \frac{\partial^2}{\partial x^2}(\lambda(D)D)$ where $\lambda(D) = 1/\epsilon(E(D))$; some properties of solutions of initial-value problems for this latter equation, with compactly supported initial data, are also derived.

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NONEXISTENCE OF SMOOTH ELECTROMAGNETIC FIELDS IN NONLINEAR DIELECTRICS[1]
II. SHOCK DEVELOPMENT IN A HALF-SPACE.

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1. Introduction

We study in this paper the propagation of an electromagnetic wave into an unbounded domain occupied by a nonlinear dielectric substance. In some Cartesian coordinate system (x_i) on the domain $\Omega \subsetneq R^3$ we assume that the domain consists of the half-space $x_1 > 0$; the propagating electromagnetic wave is assumed to be of the form

$$(1.1) \quad \underset{\sim}{E} = (0, E_2(x_1,t), 0), \quad \underset{\sim}{H} = (0, 0, H_3(x_1,t))$$

and the properties of the nonlinear dielectric substance occupying Ω are delineated by the nonlinear constitutive relations

$$(1.2) \quad \underset{\sim}{D} = \varepsilon(\underset{\sim}{E})\underset{\sim}{E}, \quad \underset{\sim}{B} = \mu(\underset{\sim}{H})\underset{\sim}{H}$$

which determine, respectively, the electric displacement and magnetic field in Ω in terms of the electric field and magnetic intensity. The constitutive relations (1.2) are commonly considered in work on nonlinear optics [1], [2] with $\varepsilon > 0$, $\mu > 0$ being scalar-valued functions of their respective fields. As $\underset{\sim}{D} = \varepsilon_0 E + \underset{\sim}{P}(\underset{\sim}{E})$ where $\underset{\sim}{P}$ is the macroscopic polarization vector and ε_0 is the permittivity of free space, and $\underset{\sim}{P}(\underset{\sim}{E}) = \chi(\underset{\sim}{E})\underset{\sim}{E}$, where $\chi(\underset{\sim}{E})$ is the susceptibility, $\underset{\sim}{D} = (\varepsilon_0 + \chi(\underset{\sim}{E}))\underset{\sim}{E}$. For an isctropic material, $\chi = \chi(\|\underset{\sim}{E}\|)$, $\mu = \mu(\|\underset{\sim}{H}\|)$ and, in most treatments in nonlinear optics texts these functions are expanded in series of the form

$$(1.3) \quad \begin{cases} \chi = \chi_0 + \chi_1 E + O(E^2), & E = \|\underset{\sim}{E}\| \\ \mu = \mu_0 + \mu_1 H + O(H^2), & H = \|\underset{\sim}{H}\| \end{cases}$$

where μ_0 is the permeability of free space, χ_0 is the linear susceptibility, χ_1 the first nonlinear susceptibility, and so forth. Experimentally $\chi_1 \ll \chi_0$ so that the presence of the term of first order in E in the expression for χ is of most interest when E is large (e.g. in a laser beam) while for most purposes one may assume that $\mu = \mu_0$. Many of our results apply to materials in which $\mu(\underset{\sim}{H})$ and $\varepsilon(\underset{\sim}{E}) = \varepsilon_0 + \chi(\underset{\sim}{E})$ are more general than the forms implied by (1.3).

In addition to the forms (1.1), (1.2), respectively, for the electromagnetic wave propagating in Ω, and the constitutive relations which delineate the dielectric material that occupies Ω, we need Maxwell's equations which we take in the form

$$(1.4) \quad \begin{cases} \dfrac{\partial \underset{\sim}{B}}{\partial t} = -\nabla \times \underset{\sim}{E}, & \operatorname{div} \underset{\sim}{B} = 0 \\[2ex] \dfrac{\partial \underset{\sim}{D}}{\partial t} = \nabla \times \underset{\sim}{H}, & \operatorname{div} \underset{\sim}{D} = 0 \end{cases}$$

We also introduce the notation

$$\tilde{\varepsilon}(\zeta) = \varepsilon((0,\zeta,0)), \quad \tilde{\mu}(\zeta) = \mu((0,0,\zeta)), \quad \zeta \in R^1$$

and assume that

h(i) $\tilde{\varepsilon}(\cdot), \tilde{\mu}(\cdot) \in C^1(R^1)$

h(ii) $\tilde{\varepsilon}(\zeta) \neq 0, \quad \tilde{\mu}(\zeta) \neq 0, \quad \forall \zeta \in R^1$

h(iii) $(\zeta\tilde{\varepsilon}(\zeta))' > 0, \quad (\zeta\tilde{\mu}(\zeta))' > 0, \quad \forall \zeta \in R^1$ or, at least, for all ζ with $|\zeta|$ sufficiently small.

From (1.2) and the assumed form of the wave (1.1) we obtain

$$\underset{\sim}{D} = (0, D_2(x_1,t), 0), \quad \underset{\sim}{B} = (0,0,B_3(x_1,t))$$

with

$$(1.5) \quad \begin{aligned} D_2(x_1,t) &= \tilde{\varepsilon}(E_2(x_1,t))E_2(x_1,t) \\ B_3(x_1,t) &= \tilde{\mu}(H_3(x_1,t))H_3(x_1,t) \end{aligned}$$

By hypotheses h(III) these relations may be inverted so as to yield

$$(1.6) \quad \begin{cases} E_2(x_1,t) = \dfrac{1}{\tilde{\varepsilon}(E(D_2(x_1,t))} D_2(x_1,t) \equiv \lambda(D_2(x_1,t))D_2(x_1,t) \\[2ex] H_3(x_1,t) = \dfrac{1}{\tilde{\mu}(H(B_3(x_1 t))} B_3(x_1,t)) \equiv \gamma(B_3(x_1,t))B_3(x_1,t) \end{cases}$$

where $\forall \zeta \in R^1$ we have defined

$$\lambda(\zeta) = \frac{1}{\tilde{\varepsilon}(E(\zeta))}, \quad \gamma(\zeta) = \frac{1}{\tilde{\mu}(H(\zeta))}$$

with (for $\rho, \zeta \in R^1$)

$$\begin{cases} \rho = \tilde{\varepsilon}(\zeta)\zeta \to \zeta = E(\rho) = E(\tilde{\varepsilon}(\zeta)\zeta) \equiv \lambda(\rho)\rho \\ \rho = \tilde{\mu}(\zeta)\zeta \to \zeta = H(\rho) = H(\tilde{\mu}(\zeta)\zeta) \equiv \gamma(\rho)\rho \end{cases}$$

Therefore $\forall \rho \in R^1$ (or $\forall \rho$ with $|\rho|$ sufficiently small)

$$(1.7) \quad (\rho\lambda(\rho))' \equiv \frac{d\zeta}{d\rho} = \frac{dE(\rho)}{d\rho} = \frac{1}{d\rho/d\zeta} > 0$$

by h(iii) and, similarly, $(\rho\gamma(\rho))' > 0$, $\forall\rho \in R^1$. Note that in the special case where $\mu = \mu_0$, $\gamma = 1/\mu_0 > 0$. In the next section we demonstrate that shock waves may be expected to form in the electromagnetic wave (1.1) as it propagates into the half-space $x_1 > 0$; We also derive some estimates for the maximal time of existence of a C^1 wave and estimate the distance travelled by the wave into the half-space until the development of the shock; these latter results are presented in §3.

There is considerable literature, ref. [10]-[17], on shock development in electromagnetic theory with the papers that are closest to the present work, in spirit, being those of Broer [10], [11], Katayev [16], and Jeffrey [11], [12]. The work of Broer, however, is not applicable to those important situations in which the quasilinear evolution equations are not genuinely nonlinear while our work differs from that of Jeffrey by virtue of the fact that by working with the constitutive relations (1.2) and employing $\underset{\sim}{D}$ and $\underset{\sim}{B}$ as our basic variables, instead of $\underset{\sim}{E}$ and $\underset{\sim}{B}$, as in [12], or $\underset{\sim}{E}$ and $\underset{\sim}{H}$, as in [11], we are able to write our evolution system in conservation form; the importance of working with $\underset{\sim}{D}$ instead of $\underset{\sim}{E}$ was emphasized in [6] and is based on the fact that $\underset{\sim}{E}$ is, in general, not divergence free in a nonlinear dielectric while $\underset{\sim}{D}$ is if there is zero free charge. Some of the consequences of having the equations for $\underset{\sim}{D}$, $\underset{\sim}{B}$ in conservation form, i.e., the implications of the Rankine-Hugoniot and Lax k-shock conditions are developed in the next section. The application of Lax's elegant work [3], also simplifies the asymptotic estimate for t_{max} in §2 and makes possible a rather simple and explicit computation in §3 for S_{max} the distance travelled by the wave into the half-space before shock development occurs.

2. Shock Development and Propagation

To simplify the notation in this section we set $x = x_1$, $D = D_2$, $E = E_2$, $H = H_3$, $B = B_3$. In view of the forms of the electromagnetic field vectors in the wave entering the half space $x_1 > 0$ Maxwell's equations (1.4) reduce to the pair of equations

$$(2.1) \qquad \frac{\partial D}{\partial E} \cdot \frac{\partial E}{\partial t} = -\frac{\partial H}{\partial x}\,; \quad \frac{\partial B}{\partial H} \cdot \frac{\partial H}{\partial t} = -\frac{\partial E}{\partial x}$$

or

$$(2.2) \qquad \begin{cases} (\tilde{\varepsilon}(E)E)' E_t + H_x = 0 \\ (\tilde{\mu}(H)H)' H_t + E_x = 0 \end{cases}$$

Setting, $a(\zeta) = \dfrac{1}{(\tilde{\varepsilon}(\zeta)\zeta)'} > 0$, $b(\zeta) = \dfrac{1}{(\tilde{\mu}(\zeta)\zeta)'} > 0$, $\forall \zeta \in R^1$, we see that $E(x,t)$, $H(x,t)$ satisfy the first-order quasilinear system

$$(2.3) \qquad \begin{cases} E_t + a(E)H_x = 0 \\ H_t + b(H)E_x = 0 \end{cases}$$

which is, unfortunately, not in the usual conservation form. We thus rewrite the system in the form

$$\frac{E_t}{a(E)} + H_x = 0, \quad \frac{H_t}{b(H)} + E_x = 0$$

and note that in view of the definitions of $a(\cdot)$, $b(\cdot)$, and (1.5)

$$(2.4) \qquad D(x,t) = \int_0^{E(x,t)} \frac{d\xi}{a(\xi)}\ , \quad B(x,t) = \int_0^{H(x,t)} \frac{d\xi}{b(\xi)}$$

Clearly, $D_t(x,t) = \frac{1}{a(E(x,t))} E_t(x,t)$, $B_t(x,t) = \frac{1}{b(H(x,t))} H_t(x,t)$ and as $a(\xi) > 0$, $b(\xi) > 0$ the relations (2.4) are invertible with, in fact, $E(x,t) = \mathcal{E}(D(x,t))$ and $H(x,t) = \mathcal{H}(B(x,t))$. Therefore, the system (2.3) is equivalent to the first-order quasilinear system

$$(2.5) \qquad \begin{cases} D_t + \mathcal{H}'(B)B_x = 0 \\ B_t + \mathcal{E}'(D)D_x = 0 \end{cases}$$

where $\mathcal{H}(B) = \gamma(B)B$, $\mathcal{E}(D) = \lambda(D)D$. If we rewrite (2.5) as

$$(2.6) \qquad \begin{pmatrix} D \\ B \end{pmatrix}_{,t} + \begin{pmatrix} 0 & \mathcal{H}'(B) \\ \mathcal{E}'(D) & 0 \end{pmatrix} \begin{pmatrix} D \\ B \end{pmatrix}_{,x} = 0$$

then, clearly, the system for D, B is in the usual conservation form

$$\frac{\partial}{\partial t} u_i + \frac{\partial}{\partial x} f_i = 0, \quad i = 1,2$$

where $\vec{u} = \begin{pmatrix} D \\ B \end{pmatrix}$ and $\vec{f} = \begin{pmatrix} \mathcal{H}(B) \\ \mathcal{E}(D) \end{pmatrix}$. In the most common situation, that of a nonmagnetic material, $\mu(H) = \mu_0$ so that $\mathcal{H}'(B) = 1/\mu_0$ and (2.6) reduces to

$$(2.7) \qquad \begin{pmatrix} D \\ B \end{pmatrix}_{,t} + \begin{pmatrix} 0 & 1/\mu_0 \\ \mathcal{E}'(D) & 0 \end{pmatrix} \begin{pmatrix} D \\ B \end{pmatrix}_{x} = 0$$

With (2.7) we associate initial data of the form

$$(2.8) \qquad D(x,0) = D_0(x), \quad B(x,0) = B_0(x)$$

and real characteristics in the x,t plane

$$(2.9) \qquad \frac{dx}{dt} = \pm \sqrt{\frac{E'(D(x,t))}{\mu_0}}$$

Note that for $\zeta \in R^1$. $E'(\zeta) = (\zeta\lambda(\zeta))' > 0$ by (1.7). The positivity of $E'(\zeta)$, $\forall\zeta \in R^1$ is equivalent to the strict hyperbolicity of the system (2.7); via standard a priori estimates (Lax [3], Nishida [4]) on the Riemann Invariants associated with the system (2.7) (these being defined below) it can be shown that if $\sup_{R^1}|D_0(x)|$ and $\sup_{R^1}|B_0(x)|$ are sufficiently small, and $E'(0) > 0$, then for as long as a sufficiently smooth solution $(D(x,t), B(x,t))$ of (2.7), (2.8) exists, on $0 \le t < t_{max}$ for instance, we will have $E'(D(x,t)) > 0$, $x \in R^1$, $0 \le t < t_{max}$. *For sufficiently small initial data*, therefore, the (real) characteristics (2.9) are well defined on the maximal interval of existence of C^1 solution even if only local hyperbolicity of (2.7) obtains, i.e., even if we only have $E'(0) > 0$.

We now define the Riemann Invariants associated with (2.7) to be

$$(2.10) \qquad \begin{cases} r(D,B) = B + \dfrac{1}{\sqrt{\mu_0}} \displaystyle\int_0^D \sqrt{E'(\zeta)}\, d\zeta \\[2ex] s(D,B) = B - \dfrac{1}{\sqrt{\mu_0}} \displaystyle\int_0^D \sqrt{E'(\zeta)}\, d\zeta \end{cases}$$

By standard results r and s are constant along their respective characteristics, i.e.,

$$(2.11)\quad \begin{cases} r'(x,t) \equiv \dfrac{\partial r}{\partial t} - \sqrt{\dfrac{E'(D(x,t))}{\mu_0}}\ \dfrac{\partial r}{\partial x} = 0 \\[2ex] s'(x,t) \equiv \dfrac{\partial s}{\partial t} + \sqrt{\dfrac{E'(D(x,t))}{\mu_0}}\ \dfrac{\partial s}{\partial x} = 0 \end{cases}$$

and we have the following blow-up results which are consequences, respectively, of the work of Lax [3] and Klainerman and Majda [5].

(A) If $D_0(x)$ is periodic on R^1, $B_0(x) \equiv 0$, and $E''(0) \neq 0$ (so that the problem exhibits genuine nonlinearity) then finite-time blow-up must occur for

$$r_x(x,t) \equiv B_x(x,t) + \frac{1}{\sqrt{\mu_0}} \sqrt{E'(D(x,t))}\, D_x(x,t)$$

$$= -\mu_0 D_t(x,t) + \frac{1}{\sqrt{\mu_0}} \sqrt{E'(D(x,t))}\, D_x(x,t)$$

$\rightarrow \nabla_{(x,t)}D \equiv (D_t, D_x)$ must blow-up in finite time and a shock develops. Futhermore, it is a consequence of the work in [3] that

$$(2.12)\qquad t_{max} \simeq \frac{\mu_0}{\max|D_0'(x)|} \cdot \frac{\sqrt{E'(0)}}{|E''(0)|}$$

a result we will apply in §3 to a specific class of nonlinear dielectric materials.

(B) Suppose that $D_0(x)$, $B_0(x)$ both have compact support in R^1; then so will

$$\begin{cases} r(x,0) = B_0(x) + \dfrac{1}{\sqrt{\mu_0}} \displaystyle\int_0^{D_0(x)} \sqrt{E'(\zeta)}\, d\zeta \\[2ex] s(x,0) = B_0(x) - \dfrac{1}{\sqrt{\mu_0}} \displaystyle\int_0^{D_0(x)} \sqrt{E'(\zeta)}\, d\zeta \end{cases}$$

From the recent work of Klainerman and Kajda [5] it then follows that if $r(\cdot,0)$, $s(\cdot,0)$ are also of class C^1 then any C^1 solution of the initial-value problem for the diagonalized system (2.11) must develop singularities in finite time in the first derivatives r_x, s_x if $E'(\zeta)$ is not constant on any open interval. For example, if with $\lambda_0 > 0$, $\lambda(\zeta) = \lambda_0 + \lambda_2\zeta^2$ then $E(\zeta) = \lambda_0\zeta + \lambda_2\zeta^3$. Clearly $E'(0) = \lambda_0 > 0$ (local hyperbolicity) and $E''(0) = 0$ (loss of genuine nonlinearity) but $E''(\zeta) = 6\lambda_2\zeta \neq 0$, if $\zeta \neq 0$, so that the result of [5] applies for C^1, compactly supported initial data.

Remarks. If (D,B) is a sufficiently smooth solution of (2.7), say class C^2, then clearly we may eliminate so as to obtain the scalar nonlinear wave equation for $D(x,t)$:

$$(2.13) \qquad \frac{\partial^2}{\partial t^2} D(x,t) = 1/\mu_0 \frac{\partial^2}{\partial x^2} E(D(x,t)).$$

This equation was derived by this author in [6] by specializing the three dimensional evolution equations

$$(2.14) \qquad \mu_0 \frac{\partial^2 D_i}{\partial t^2} = \nabla^2(\lambda(\underset{\sim}{D})D_i) - \mathrm{grad}_i(\mathrm{grad}\lambda(D)\cdot D),$$

obtained under the assumption that $\underset{\sim}{B} = \mu_0\underset{\sim}{H}$, $\underset{\sim}{E} = \lambda(\underset{\sim}{D})\underset{\sim}{D}$ in Ω, to the case of an electromagnetic wave of the form (1.1) propagating through an ∞ nonlinear dielectric cylinder, with the direction of propagation directed along the axis of the cylinder. Several facts may be noted about the simple wave equation (2.13)

1) Suppose we set $D(x,t) = G_x(x,t)$ in (2.13). Then provided that $G(x,t)$ is sufficiently smooth we obtain

$$G_{ttx} = 1/\mu_0 E(G_x)_{xx} \rightarrow$$

$$G_{tt} = 1/\mu_0 E(G_x)_x + K(t)$$

where $K(t)$ is an arbitrary function of t. Thus the usual scalar nonlinear wave equation which arises in one-dimensional motions of a nonlinear elastic body is not equivalent to (2.13).

2) Suppose that $D(x,t)$ is a solution of class C^2 of an initial-value problem on R^1 associated with (2.13) and is such that for $0 \le t < t_{max}$

$$(2.15) \qquad \int_{-\infty}^{t} E(D(-\infty,t))_x dt < \infty \quad \underline{\text{where}}$$

$$E(D(-\infty,t))_x \equiv \lim_{x\to-\infty} [\frac{\partial}{\partial x} E(D(x,t))]$$

If we set

$$(2.16) \qquad \hat{B}(x,t) = -\mu_0 \int_{-\infty}^{x} D_t(y,t)dy - \int_{-\infty}^{t} E(D(-\infty,t))_x dt + C$$

where C is an arbitrary constant, then

$$\hat{B}_x(x,t) = -\mu_0 D_t(x,t), \text{ and by (2.13)}$$

$$\hat{B}_t(x,t) = -\mu_0 \int_{-\infty}^{x} \frac{\partial^2}{\partial t^2} D(y,t)\, dy - E(D(-\infty,t))_x$$

$$= - \int_{-\infty}^{x} \frac{\partial^2}{\partial y^2} E(D(y,t))dy - E(D(-\infty,t))_x$$

$$= - E(D(x,t))_x$$

so that the pair $(D(x,t), \hat{B}(x,t))$ is a solution of the system (2.7). This equivalence between the system (2.7) and the scalar nonlinear wave equation (2.13) only holds if $\hat{B}$ is well-defined, i.e., if (2.15) obtains. As

$$E(D(-\infty,t))_x = \lim_{x\to-\infty} [E'(D(x,t))D_x(x,t)],$$

if (2.13) is strictly hyperbolic, i.e., $E'(\zeta) > 0$, $\forall\zeta \in R^1$, then clearly (2.15) will obtain if $\lim_{t\to-\infty} |D_x(x,t)| = 0$, $0 \le t < t_{max}$. We now return to the system (2.7); shock development and propagation in the more general system (2.5), which is strictly hyperbolic provided $E'(\zeta)H'(\zeta) > 0$, $\forall\zeta \in R^1$, will be dealt with in a forthcoming paper [7]. If we let s denote the speed of the shock which develops either in case (A) or case (B) above, and [F] the jump in quantity F across the shock then the Rankine-Hugoniot conditions require that $s[u_k] = [f_k]$, $k = 1,2$ where

$$\vec{u} = \begin{pmatrix} D \\ B \end{pmatrix} \text{ and } \vec{f} = \begin{pmatrix} 1/\mu_0\ B \\ \lambda(D)D \end{pmatrix}$$

We therefore obtain the conditions

$$(2.17)\quad \begin{cases} s[D] = \dfrac{1}{\mu_0}[B] = [H] \\ s[B] = [\lambda(D)D] = [E] \end{cases}$$

from which follow the simple relations

$$\text{(i)}\quad [D][E] = [H][B] = \frac{1}{\mu_0}[B]^2$$

$$\text{(ii)}\quad s^2[D] = \frac{s}{\mu_0}[B] = \frac{1}{\mu_0}[E]$$

The shock speed is, therefore, given by

$$(2.18)\quad s = \pm\frac{1}{\sqrt{\mu_0}}\sqrt{\frac{[E]}{[D]}} = \pm\frac{1}{\sqrt{\mu_0}}\sqrt{\frac{[\lambda(D)D]}{[D]}}\,.$$

so that two shocks are possible, one moving to the left and one moving to the right. We now apply Lax's [8], [9] k-shock conditions to the system $\frac{\partial}{\partial t}u_i + \frac{\partial}{\partial x}f_i = 0$, $i = 1,2$; i.e., we require that for either $k = 1$ or $k = 2$

$$(2.19)\quad \lambda_K(\vec{u}_-) > s > \lambda_k(\vec{u}_+)$$

where $\vec{u} = \begin{pmatrix} D \\ B \end{pmatrix}$, $\vec{f}(\vec{u}) = \begin{pmatrix} 1/\mu_0 B \\ \lambda(D)D \end{pmatrix}$, and the λ_k, $k = 1,2$ are the distinct real eigenvalues of the matrix $\nabla_{\vec{u}}\vec{f}$. However, by (2.9)

$$(2.20)\quad \lambda_1 = \frac{1}{\sqrt{\mu_0}}\sqrt{E'(D)}\,,\quad \lambda_2 = -\frac{1}{\sqrt{\mu_0}}\sqrt{E'(D)}$$

In (2.19), $\vec{u}_- = \begin{pmatrix} D_- \\ B_- \end{pmatrix}$, $\vec{u}_+ = \begin{pmatrix} D_+ \\ B_+ \end{pmatrix}$ denote, respectively, the values of $\vec{u}$ behind and in front of the shock. The conditions (2.19) represent one formulation of a classical entropy condition for solutions of hyperbolic conservation laws containing a shock.

Using (2.20), (2.19) becomes

$$(2.21)\begin{cases} \sqrt{E'(D_-)} > \sqrt{\mu_0}\, s > \sqrt{E'(D_-)} & \text{or} \\ -\sqrt{E'(D_-)} > \sqrt{\mu_0}\, s > -\sqrt{E'(D_+)} & \end{cases}$$

However, by the definition of E, $E = E(D) = E(D(E))$, where $D(E) = \tilde{\varepsilon}(E)E$ so that (2.21) is equivalent to

$$(2.22)\begin{cases} \dfrac{1}{\sqrt{D'(E_-)}} > \sqrt{\mu_0}\, s > \dfrac{1}{\sqrt{D'(E_+)}} & \text{or} \\ -\dfrac{1}{\sqrt{D'(E_-)}} > \sqrt{\mu_0}\, s > \dfrac{1}{\sqrt{D'(E_+)}} & \end{cases}$$

For the shock moving to the right, $s_r = \dfrac{1}{\sqrt{\mu_0}} \sqrt{\dfrac{[E]}{[D(E)]}}$ so that only the first inequality in (2.22) makes sense, and we must have

$$(2.23a)\qquad \sqrt{D'(E_-)} < \sqrt{\frac{[D(E)]}{[E]}} < \sqrt{D'(E_+)}$$

For the shock moving to the left $s_\ell = -\dfrac{1}{\sqrt{\mu_0}} \sqrt{\dfrac{[E]}{[D(E)]}}$ so that only the second inequality in (2.22) makes sense, and we obtain

$$(2.23b)\qquad \sqrt{D'(E_-)} > \sqrt{\frac{[D(E)]}{[E]}} > \sqrt{D'(E_+)}$$

Thus the k-shock conditions of Lax predict that (2.23a) must hold for the shock moving to the right while (2.23b) must hold for the shock moving to the left. We now examine the implications of (2.23a,b) for the simple but physically important case where

$$(2.24) \qquad \tilde{\varepsilon}(E) = \varepsilon_0 + \varepsilon_2 E^2, \quad \varepsilon_0 > 0, \ \varepsilon_2 > 0$$

We recall that by the definitions of E, $\mathcal{D}$, $\forall \zeta \in R^1$, $\zeta = E(\mathcal{D}(\zeta))$ where $\mathcal{D}(\zeta) = \tilde{\varepsilon}(\zeta)\zeta \equiv \varepsilon_0\zeta + \varepsilon_2\zeta^3$. Thus $\mathcal{D}(\zeta) = 0$ if and only if $\zeta = 0$. A direct computation yields

$$(2.25) \qquad E''(\mathcal{D}(\zeta)) = -\frac{E'(\mathcal{D}(\zeta))\mathcal{D}''(\zeta)}{\mathcal{D}'^2(\zeta)}$$

$$\text{where} \quad E'(\mathcal{D}(\zeta)) = \left.\frac{dE}{d\mathcal{D}}\right|_{\zeta}, \quad \mathcal{D}'(\zeta) = \frac{d\mathcal{D}}{d\zeta}$$

If $E'(\mathcal{D}(\zeta)) > 0$ (at least for $|\zeta|$ sufficiently small) then

$$(2.26) \qquad E''(\mathcal{D}(\zeta)) = -E'(\mathcal{D}(\zeta)) \cdot \left[\frac{6\varepsilon_2\zeta}{(\varepsilon_0+3\varepsilon_2\zeta^2)^2}\right]$$

so that $E''(\mathcal{D}(0)) = E''(0) = 0$ but $E''(\mathcal{D}(\zeta)) \neq 0$ for all $\zeta \neq 0$. Thus for C^1 initial data with compact support the results of Klainerman and Majda [5], as previously stated, apply and shocks will develop in finite time. As $\mathcal{D}'(E) = \varepsilon_0 + 3\varepsilon_2 E^2$, the condition (2.32a) relative to the shock moving to the right with speed

$$s_r = \frac{1}{\sqrt{\mu_0}}\sqrt{\frac{[E]}{[\varepsilon_0 E + \varepsilon_2 E^3]}}$$

$$= \frac{1}{\sqrt{\mu_0}}\sqrt{\frac{E_+ - E_-}{\varepsilon_0(E_+ - E_-) + \varepsilon_2(E_+^3 - E_-^3)}}$$

$$= \mu_0^{-1/2}(\varepsilon_0+\varepsilon_2(E_+^2 +(E_++E_-)E_-))^{-1/2}$$

is

$$(2.27) \quad \sqrt{\varepsilon_0+3\varepsilon_2 E_-^2} < \sqrt{\varepsilon_0+\varepsilon_2(E_+^2+(E_++E_-)E_-)}$$

$$< \sqrt{\varepsilon_0+3\varepsilon_2 E_+^2}$$

This last inequality clearly implies that

$$(2.28a) \quad \begin{cases} E_+^2 + E_+E_- > 2E_-^2 \\ E_-^2 + E_+E_- < 2E_+^2 \end{cases}$$

from which we easily deduce that

$$(2.28b) \quad 2E_+^2 - E_-^2 > E_+E_- > 2E_-^2 - E_+^2$$

and

$$(2.28c) \quad E_+^2 > E_-^2 \quad \text{(across the shock moving to the right)}$$

In a completely analogous fashion we obtain, for the shock moving to the left

$$s_\ell = -\mu_0^{-1/2}(\varepsilon_0+\varepsilon_2(E_+^2 +(E_+ + E_-)E_-))^{-1/2}$$

while across this shock

$$(2.29a) \quad \begin{cases} 2E_-^2 > E_+^2 + E_+E_- \\ 2E_+^2 < E_-^2 + E_+E_- \end{cases}$$

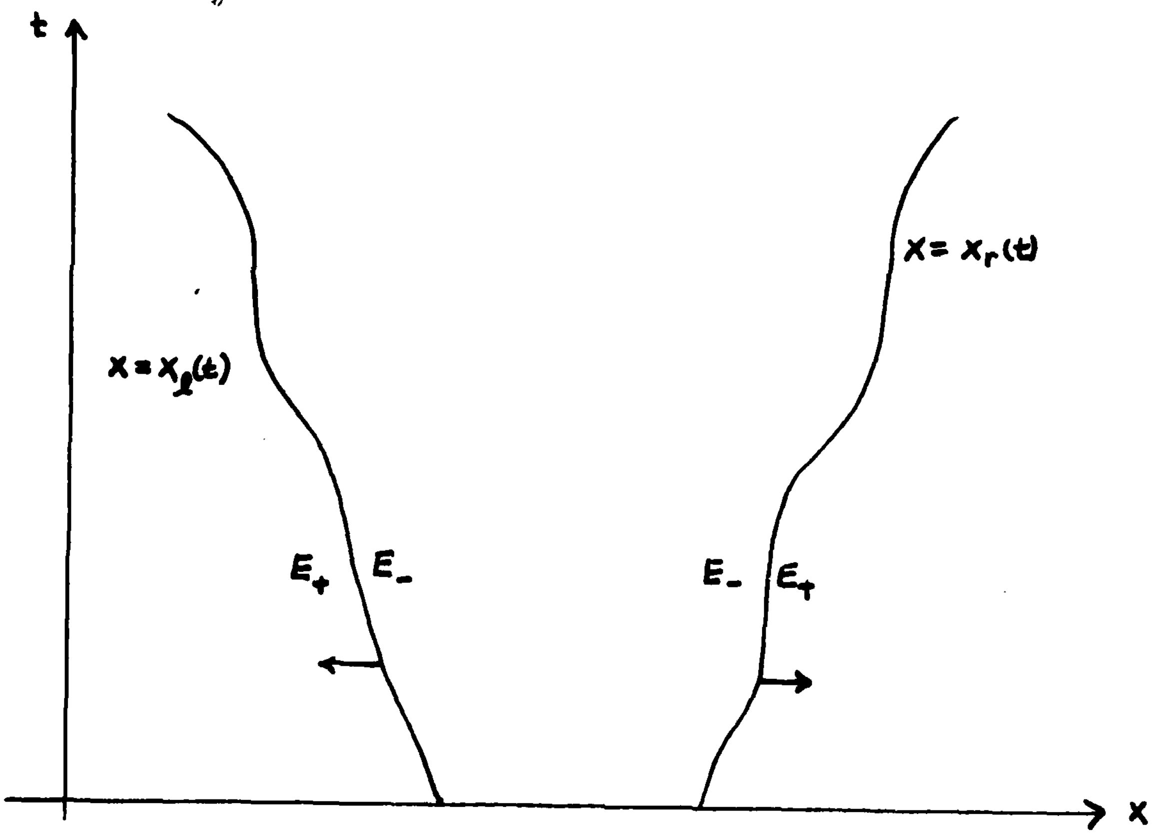
from which we deduce that

$$(2.29b) \qquad 2E_-^2 - E_+^2 > E_+E_- > 2E_+^2 - E_-^2$$

and

$$(2.29c) \qquad E_-^2 > E_+^2 \quad \text{(across the shock moving to the left.)}$$

The situation corresponding to (2.28c) and (2.29c) is depicted below where we have denoted the shock moving to the right with speed s_r by $x = x_r(t)$ and the shock moving to the left with speed s_ℓ by $x = x_\ell(t)$



In the figure sketched above E_+, E_- denote, respectively, the values of $E(x,t)$ in front of and behind the respective shocks. For the shocks $x_\ell(t)$, $x_r(t)$ therefore, (2.28c) and (2.29c) predict that

$E^2(x,t)$ must increase as we cross the respective shocks, moving in the direction of increasing x. However, that part of the energy residing in the electromagnetic wave which depends on $\mathcal{E}$ is given by $\mathfrak{E}_{\mathcal{E}} = \frac{1}{2}\mathcal{D}(\mathcal{E}) \cdot \mathcal{E}$ or

$$\mathfrak{E}_{\mathcal{E}} = \tfrac{1}{2}\mathcal{D}(E)E = \tfrac{1}{2}((\varepsilon_0+\varepsilon_2 E^2)E)E$$

and thus we have the result that

$$(2.30) \qquad \mathfrak{E}_{\mathcal{E}}(x,t) \equiv \tfrac{1}{2}\varepsilon_0 E^2(x,t) + \tfrac{1}{2}\varepsilon_2 E^4(x,t)$$

must increase as we move across the respective shocks in the direction of increasing x.

<u>Remarks</u>. For the equations (2.3), i.e.

$$(2.3) \qquad \begin{cases} E_t + a(E)H_x = 0 \\ H_t + b(H)E_x = 0 \end{cases}$$

where $a(E) > 0$, $b(H) > 0$ it is a simple matter to show that there exist solutions of the form $E(x,t) = E_0(x-\lambda t)$, $H(x,t) = H\cdot(x-\lambda t)$ where $E_0(x) = E(x,0)$, $H_0(x) = H(x,0)$. In fact $\lambda = \pm\sqrt{a(E)b(H)}$ so that (implicit) travelling wave solutions of the form

$$(2.31) \qquad \begin{cases} E = E_0(x \pm \sqrt{a(E)b(H)}\ t) \\ H = H_0(x \pm \sqrt{a(E)b(H)}\ t) \end{cases}$$

may be well-defined, at least for small values of t.[2] If the material is such that $\tilde{\mu}(H) = \mu_0$ then $b(H) = \mu_0^{-1}$ and (2.31)

[2] Solutions of the form (2.31), for the system (2.3) have been discussed by Katayev [16].

assumes the form

$$(2.32)\quad \begin{cases} E = E_0(x \pm \mu_0^{-\frac{1}{2}} \sqrt{a(E)}\, t) \\ H = H_0(x \pm \mu_0^{-\frac{1}{2}} \sqrt{a(E)}\, t) \end{cases}$$

If we define

$$F(x,t,E) \equiv E - E_0(x \pm \mu_0^{-\frac{1}{2}} \sqrt{a(E)}\, t)$$

then $F(x_0,0,0) = 0$, for any value of x_0 such that $E_0(x_0) = 0$, and

$$F_E(x,t,E) = 1 \pm \tfrac{1}{2}\mu_0^{-\frac{1}{2}} E_0'(x \pm \mu_0^{-\frac{1}{2}}\sqrt{a(E)}t)\cdot(a(E))^{-\frac{1}{2}} a'(E)$$

$$(2.33)\quad \begin{cases} a(E) = \tilde{\varepsilon}(E) + \tilde{\varepsilon}'(E)E \to a(0) = \tilde{\varepsilon}(0) \\ a'(E) = 2\tilde{\varepsilon}'(E) + \tilde{\varepsilon}''(E)E \to a'(0) = 2\tilde{\varepsilon}'(0). \end{cases}$$

In the typical situation where $\tilde{\varepsilon}(E) = \varepsilon_0 + \varepsilon_2 E^2$, $\varepsilon_0 > 0$, $\varepsilon_2 > 0$, $a(0) = \varepsilon_0$ and $a'(0) = 0$ so that $F_E(x_0,0,0) = 1$. If $\tilde{\varepsilon}(E) = \varepsilon_0 + \varepsilon_1 E$ then $a(0) = \varepsilon_0$, $a'(0) = 2\varepsilon_1$ and $F_E(x_0,0,0) = 1 \pm \frac{\varepsilon_1 \mu_0^{-\frac{1}{2}}}{\varepsilon_0^{\frac{1}{2}}} E_0'(x_0)$. By the implicit function theorem, therefore, if x_0 is such that $E_0(x_0) = 0$ and, either,

(i) $\tilde{\varepsilon}(E) = \varepsilon_0 + \varepsilon_2 E^2$

or

(ii) $\tilde{\varepsilon}(E) = \varepsilon_0 + \varepsilon_1 E$, and, $\varepsilon_0^{\frac{1}{2}} \pm \varepsilon_1 \mu_0^{-\frac{1}{2}} E_0'(x_0) \neq 0$

a solution $E = E(x,t)$ of $F(x,t,E) = 0$ will exist for $|x-x_0|$

and $|t|$ sufficiently small.

Suppose, now, that we differentiate the first relation in (2.32) through with respect to x and solve for E_x; we easily obtain

$$(2.34) \qquad E_x = E_0'(x \pm \mu_0^{-\frac{1}{2}}\sqrt{a(E)}t) \Big/ \left(1 \neq \frac{1}{2}\sqrt{\frac{\mu_0^{-1}}{a(E)}}\, a'(E)t\right)$$

where $a(E)$, $a'(E)$ are given by (2.33). If $\exists \varepsilon^* > 0$ such that

$$(2.35) \qquad \frac{\sqrt{a(\zeta)}}{a'(\zeta)} < \frac{1}{\varepsilon^*}, \quad \forall \zeta \in R^1, \ |\zeta| \text{ sufficiently small}$$

then for the wave moving to the left with velocity $\mu_0^{-\frac{1}{2}}\sqrt{a(E)}$

$$(2.36) \qquad 1 - \frac{1}{2}\sqrt{\mu_0^{-1}} \; \frac{a'(E)}{\sqrt{a(E)}}\, t < 1 - \frac{1}{2}\sqrt{\mu_0^{-1}}\; \varepsilon^* t \to 0$$

as $t \to t^* = 2/\sqrt{\mu_0^{-1}}\, \varepsilon^*$ and, thus, by (2.34)

$$(2.37) \qquad E_x(x,t) \to +\infty \quad \text{as} \quad t \to t^*$$

It therefore follows directly from (2.34) that a shock can be expected to develop provided the constitutive relation in the material is such that (2.35) is satisfied. For the case in which $\tilde{\varepsilon}(\zeta) = \varepsilon_0 + \varepsilon_1\zeta$, $\zeta \in R^1$, (2.35) assumes the form $\sqrt{\varepsilon_0 + 2\varepsilon_1\zeta} < 2\varepsilon_1/\varepsilon^*$ which is certainly satisfied for $|\zeta|$ sufficiently small if ε^* is chosen sufficiently small; the prediction of shock development via (2.34) is not suprising in this case since we are in the genuinely nonlinear situation.

However, if $\tilde{\varepsilon}(\zeta) = \varepsilon_0 + \varepsilon_2\zeta^2$ then (2.35) is equivalent to

$$\sqrt{\varepsilon_0 + 3\varepsilon_2\zeta^2} < \frac{6\varepsilon_2}{\varepsilon^*}\,\zeta$$

which can not be satisfied $\forall\zeta$, $|\zeta|$ sufficiently small, no matter how small $\varepsilon^* > 0$ is chosen; this is, of course, a situation in which genuine nonlinearity fails and shock development can not be shown to follow directly from (2.34). This is, essentially, the sort of situation in which the utility of the result proven by Klainerman and Majda [5] becomes apparent provided the initial-value problem, formulated in terms of the fields $\underset{\sim}{B}$ and $\underset{\sim}{D}$, instead of $\underset{\sim}{E}$ and $\underset{\sim}{H}$, is such that the initial data are C^1 and compactly supported. Analogous results may be presented for the more general situation described by the system (2.3), where we do not assume, a priori, that $\tilde{\mu}(\zeta) = \mu_0$, $\forall\zeta \in R^1$, but a discussion of such results will be relegated to a forthcoming work [7]. It is noteworthy to remark that solutions of the basic form (2.32), for an electromagnetic wave propagating into a half-space filled by a nonlinear dielectric substance described by an arbitrary functional relationship between D and E, were previously obtained by Broer [10], [17] where the analysis was taken as far as the determination of Ex which, it was shown, could blow-up in finite-time thus leading to the initiation of non-uniqueness in the solution.

Remarks. If we consider a material for which $\tilde{\mu}(H) = \mu_0$ then (2.3) reduces to

$$\begin{cases} E_t + a(E)H_x = 0 \\ H_t + \dfrac{1}{\mu_0} E_x = 0 \end{cases}$$

which is easily seen to imply that $E(x,t)$ satisfies the nonlinear wave equation

$$E_{tt} - \frac{1}{\mu_0} a(E)E_{xx} = \left\{\frac{d}{dE} \ell n\ a(E)\right\} E_t^2 \tag{2.38}$$

an interesting equation in its own right which will be considered in [7].

3. Some Computational Aspects of Shock Development

In this section we derive some estimates for t_{max}, the maximal time until the development of a shock in an electromagnetic wave of the form (1.1) which is propagating into a half-space filled with a nonlinear dielectric substance. We also estimate the velocity of the wave in the nonlinear dielectric and the distance travelled by the wave into the half-space until the shock develops. We assume that the dielectric conforms to the constitutive hypothesis

$$D = \varepsilon_0 E + P(E), \quad B = \mu_0 H$$

with $P(E) = \chi_0 E + \chi_1 E^2$, $\chi_0 > 0$, $\chi_1 > 0$. The quantities χ_0, χ_1 have been defined in §1 as being, respectively, the linear and (first) nonlinear susceptibilities of the dielectric. Thus, $D = \varepsilon(E)E$, $\varepsilon(E) = \bar{\varepsilon}_0 + \bar{\varepsilon}_2 E$ with $\bar{\varepsilon}_0 = \varepsilon_0 + \chi_0$, $\bar{\varepsilon}_2 = \chi_1$. From $D = (\varepsilon_0+\chi_0)E + \chi_1 E^2$ we easily compute that

$$(3.1) \qquad E(x,t) = -\frac{(\varepsilon_0+\chi_0)}{2\chi_1} \pm \frac{1}{2\chi_1}\sqrt{(\varepsilon_0+\chi_0)^2+4\chi_1 D(x,t)}$$

Choosing the positive sign on the radical and expanding in a power series we obtain $E = \lambda(D)D + O(D^3)$ where

$$(3.2) \qquad \begin{cases} \lambda(D) = \lambda_0 + \lambda_1 D \\ \lambda_0 = \dfrac{\chi_1}{(\varepsilon_0+\chi_0)} > 0, \quad \lambda_1 = \dfrac{-\chi_1^2}{(\varepsilon_0+\chi_0)^{3/2}} < 0 \end{cases}$$

Our problem, up to terms of order D^3, is then of the form (2.7) with

$$E(D) = \frac{\chi_1}{(\varepsilon_0+\chi_0)} D - \frac{\chi_1^2}{(\varepsilon_0+\chi_0)^{3/2}} D^2$$

so that

$$(3.3) \quad \begin{cases} E'(D) = \dfrac{\chi_1}{(\varepsilon_0+\chi_0)} - \dfrac{2\chi_1^2}{\left(\varepsilon_0+\chi_0\right)^{3/2}} D > 0 \quad \text{for} \\ \\ |D| < (\varepsilon_0+\chi_0)^{\frac{1}{2}}/2\chi_1 \end{cases}$$

We assume that initial fields of the form (2.8) are prescribed with $B_0(x) \equiv 0$, $D_0(x)$ periodic on R^1, and sufficiently small, so that the standard a priori estimates, based on the Riemann invariants associated with (2.7), imply that (3.3) is satisfied for as long as a C^1 field $D(x,t)$ exists. Actually

$$D_0(x) = (\varepsilon_0+\chi_0)E_0(x) + \chi_1 E_0^2(x)$$

so that

$$(3.4) \quad \begin{aligned} D_0'(x) &= (\varepsilon_0+\chi_0)E_0'(x)+2\chi_1 E_0'(x)E_0(x) \\ &= E_0'(x)[(\varepsilon_0+\chi_0)+2\chi_1 E_0(x)] \\ &\approx 2\chi_1 E_0(x)E_0'(x) \end{aligned}$$

In making the approximation in (3.4) we are assuming that the initial field is that of a high intensity (laser) beam whose strength is of the order of magnitude 10^9(volts/meter) while ε_0, the permittivity of free space is of the order 10^{-13}(Columbs/Newton-Meters^2); χ_0 is of the order 10^{-13}(Columbs/Newton-Meters^2), and χ_1 is of the order 10^{13}(Columbs/volts^2). For example, $\chi_1 = 4 \times 10^{-13}$ for index matched KDP [2]. The various units (MKS system) are related by the identification Newtons/Columb ≡ Volts/Meter, which are the dimensions of the electric field E (that of D, the electric induction field, as well as of P, the polarization, are then Columbs/Meters^2). Thus, in (3.4), the quantity $(\varepsilon_0+\chi_0)$ is of order 10^{-13} while $\chi_1 E_0$ is of order 10^{-4}.

Since $E''(0) = -2\chi_1^2/(\varepsilon_0+\chi_0)^{3/2} \neq 0$ the problem (2.7), (2.8) is genuinely nonlinear and the results of §2 apply. In particular, we find that

$$(3.5) \qquad t_{max} \sim \frac{\mu_0}{\max|D_0'(x)|} \frac{\sqrt{E'(0)}}{|E''(0)|}$$

$$\sim \frac{\mu_0(\varepsilon_0+\chi_0)}{\chi_1^{5/2}} (\max|E_0E_0'|)^{-1}$$

Also, if we denote by v_b the velocity of the beam in the dielectric then

$$v_b = \mu_0^{-\frac{1}{2}} \sqrt{a(E)}$$

$$= \frac{1}{\sqrt{\mu_0(\varepsilon(E)E)'}}$$

$$= \frac{1}{\sqrt{\mu_0((\varepsilon_0+\chi_0)+2\chi_1 E)}}$$

However, $\mu_0(\varepsilon_0+\chi_0)$ is of the order of magnitude 10^{-19} (μ_0, the permeability of free space is $4\pi 10^{-7} \frac{\text{Newtons-Seconds}^2}{\text{Columbs}^2}$) while $\mu_0\chi_1 E$ is of the order of magnitude 10^{-10}. Therefore,

$$\text{(c)} \qquad v_b \quad \frac{1}{\sqrt{2\mu_0 \chi_1 E}} \geq \frac{1}{\sqrt{2\mu_0 \chi_1}} (\max E)^{-\frac{1}{2}}$$

Finally, the distance travelled by the beam into the dielectric half-space until the shock develops is given approximately by

$$S_{max} \approx v_b \cdot t_{max}$$

$$\leq \frac{1}{\sqrt{2\mu_0 \chi_1}} (\max E)^{-\frac{1}{2}} \cdot \frac{\mu_0(\varepsilon_0 + \chi_0)}{\chi_1^{5/2}} (\max|E_0 E_0'|)^{-1}$$

or

$$(3.7) \qquad S_{max} \leq \frac{1}{\sqrt{2}} \frac{\mu_0^{\frac{1}{2}}(\varepsilon_0 + \chi_0)}{\chi_1^3} (\max E)^{-\frac{1}{2}} (\max|E_0 E_0'|)^{-1}$$

If over the distance travelled by the beam into the half-space, until shock development, $\max E \approx \max E_0$ then we have the (admittedly crude but, nonetheless, informative) estimate

$$(3.8) \qquad S_{max} \approx C_0 (\max E_0)^{-3/2} (\max|E_0'|)^{-1}$$

where $C_0 = \frac{1}{\sqrt{2}} \frac{\mu_0^{\frac{1}{2}}(\varepsilon_0 + \chi_0)}{\chi_1^3}$ is a characteristic material coefficient which may be associated with a particular nonlinear dielectric substance. For most common nonlinear dielectric substance, e.g., index matched KDP, C_0 will be a very large number, something of the order of magnitude 10^{21}. Thus (3.8) indicates that even for an incident high intensity beam of the order of magnitude 10^9 volts/meter a steep gradient on the incident beam will be needed so that shock development may occur within distances obtainable under laboratory conditions.

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one over the square root of 2

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